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Universal PBW-Basis of Hall–Ringel Algebras and Hall Polynomials*

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1. INTRODUCTION

The Hall–Ringel algebra of an finitary ring R (in particular a finite dimensional algebra over a finite field) is by definition in [6] a free abelian group with basis indexed by the isomorphism classes of finite R-modules and is endowed with a product by counting filtration of finite R-modules. It is an associative ring with the identity element. Such a kind of Hall–Ringel algebra was introduced by C. M. Ringel in order to investigate the connections between finite dimensional hereditary algebras of Dynkin type and simple complex Lie algebras. It turns out, see [1, 7, 8], that the Hall–Ringel algebras for finite dimensional hereditary algebras have a strong relationship with not only Kac–Moody Lie algebras but also corresponding quantized enveloping algebras. These also stimulate the study for the Hall–Ringel algebras themselves.

We find that any Hall–Ringel algebra has a nice property: its rational extension has a universal PBW basis consisting of the isoclasses of all

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indecomposable modules; this is just like the PBW theorem for the universal enveloping algebras of Lie algebras. As a consequence, the rational extension of a Hall−Ringel algebra is generated by the isoclasses of all indecomposable modules. We will give in Section 3 the definition of a universal PBW basis and the exact statement and proof of the above result.

Applying this result and the partial order introduced for its proof, we will also consider several natural questions in studying the Hall−Ringel algebras: When does a Hall−Ringel algebra equal some of its important subrings, such as the subring generated by the isoclasses of simple modules, i.e., the composition algebra; and when does a finite dimensional algebra over a finite field have Hall polynomials? In the case where R is representation-directed, Ringel has given the positive answers for all the questions above, see [6, 8]. The aim of this note is to prove them in more general settings.

Ringel proved in [7] that if R is representation-directed, then the rational extension of its Hall−Ringel algebra equals the rational extension of its composition algebra. We will show in Section 4 that this is still true if there is no short cycle in the category of all finite R-modules.

Let R be a finite dimensional algebra over a finite field. In the case where R is representation-directed or cyclic serial, Ringel in [6, 9] and Guo in [2] respectively proved that R has Hall polynomials. It has been conjectured by Ringel that any representation-finite algebra has Hall polynomials. We will show in Section 5 that if R is a representation-finite trivial extension algebra or if there is no short cycle in the category of all finite dimensional R-modules, then R has Hall polynomials.

We thank Professor C. M. Ringel; he suggested to us the name of a universal PBW basis. The first named author is indebted to Professor I. Reiten and Professor S. Smalø for the suggestion for simplifying the proofs of Theorem 4.1.

2. PRELIMINARIES

In this section, we will fix notations and recall something about representation theory of algebras and Hall−Ringel algebras. We also prove a result in homological algebra which will play a crucial role in the proofs for all of our main results.

We will always assume that R is an associative ring and have a set of idempotents $e_i \in R$ $(i \in I)$ such that $R = \oplus_{i,j \in I} e_i R e_j$. For a left R-module M, we will assume that $RM = M$ and M is of finite length, and we denote by $[M]$ the isomorphism class of M. We will denote by R-mod the category of all finite generated R-modules, by R-ind the full subcategory whose objects are representatives of $[X]$ with X indecomposable R-module, and R-fin the full subcategory of finite R-modules in R-mod. All results in this paper deal with algebras over finite fields, with all simple modules finite, and also where all the extension groups are finite.

Given a finite $\mathbf{Z}$-module M, we denote its length by $l_{\mathbf{Z}}(M)$

LEMMA 2.1. *Let*

$$\alpha : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence in R-fin. *Then for any* $M \in R$-fin,

(i) $l_{\mathbf{Z}}(\mathrm{Ext}^1_R(M, B)) \le l_{\mathbf{Z}}(\mathrm{Ext}^1_R(M, A \oplus C))$ *and we have strict inequality if there is some* $h \in \mathrm{Hom}_R(M, C)$ *which can't factor through* g;

(ii) $l_{\mathbf{Z}}(\mathrm{Ext}^l_R(B, M)) \le l_{\mathbf{Z}}(\mathrm{Ext}^1_R(A \oplus C, M))$ *and we have strict inequality if there is some* $h \in \mathrm{Hom}_R(A, M)$ *which can't factor through* f.

In particular,

$$l_{\mathbf{Z}}\big(\mathrm{Ext}^1_R(B, B)\big) \le l_{\mathbf{Z}}\big(\mathrm{Ext}^1_R(A \oplus C, A \oplus C)\big)$$

and we have strict inequality if the short exact sequence is nonsplit. As a consequence,

$$l_{\mathbf{Z}}\big(\mathrm{Ext}^1_R(B, B)\big) = l_{\mathbf{Z}}\big(\mathrm{Ext}^1_R(A \oplus C, A \oplus C)\big)$$

if and only if $B = A \oplus C$ *if and only if* α *is split.*

Proof. (i) Applying $\mathrm{Hom}_R(M, -)$ to α, we obtain a long exact sequence

$$\dots \mathrm{Hom}(M, C) \xrightarrow{\delta} \mathrm{Ext}^1_R(M, A) \to \mathrm{Ext}^1_R(M, B) \to \mathrm{Ext}^1_R(M, C) \dots .$$

So we have

$$\begin{aligned} & l_{\mathbf{Z}}\big(\mathrm{Ext}^1_R(M, B)\big) \\ &\qquad \le l_{\mathbf{Z}}\big(\mathrm{Ext}^1_R(M, A)\big) + l_{\mathbf{Z}}\big(\mathrm{Ext}^1_R(M, C)\big) - l_{\mathbf{Z}}(\mathrm{Im}\ \delta) \\ &\qquad = l_{\mathbf{Z}}\big(\mathrm{Ext}^1_R(M, A \oplus C)\big) - l_{\mathbf{Z}}(\mathrm{Im}\ \delta) \\ &\qquad \le l_{\mathbf{Z}}\big(\mathrm{Ext}^1_R(M, A \oplus C)\big). \end{aligned}$$

If there is some $h \in \mathrm{Hom}_R(M, C)$ which can't factor through g, then the above δ is not zero and hence the last inequality above must be strict.

(ii) This follows by duality.

In particular, if we take $M = B$, then

$$l_{\mathbf{Z}}\big(\mathrm{Ext}^1_R(B, B)\big) \le l_{\mathbf{Z}}\big(\mathrm{Ext}^1_R(B, A \oplus C)\big) \le l_{\mathbf{Z}}\big(\mathrm{Ext}^1_R(A \oplus C, A \oplus C)\big).$$

If the equality holds, then any morphism in $\mathrm{Hom}_R(A, A \oplus C)$ can factor through f and hence α is split. It is clear that if α is split, then the equality holds. ∎

Let $L, M_1, M_2, \ldots, M_l \in \operatorname{mod} R$. $(M_1, M_2, \ldots, M_l)$ is called a *factor sequence* of L provided there is a filtration of R-modules,

$$0 = L_l \subset L_{l-1} \subset \cdots \subset L_1 \subset L_0 = L,$$

such that $L_{i-1}/L_i \cong M_i$ for $i = 1, \ldots, l$. Denote by $F^L_{M_1, \ldots, M_l}$ the number of filtrations of M whose factor sequence is $(M_1, \ldots, M_l)$. The (integral)*Hall–Ringel algebra* $\mathcal{H}(R)$ of R is defined to be the free abelian group with a basis $\{u_{[M]}\}_{[M]}$ indexed by the isoclasses $[M]$ of finite R-modules M with the multiplication defined by

$$u_{[M]} u_{[N]} = \sum_{[L],\, L \in R\text{-fin}} F^L_{M,N} u_{[M]}.$$

Here the sum is finite since R is finitary. The *composition algebra* $\mathcal{C}(R)$ is the subalgebra of $\mathcal{H}(R)$ generated by $\{u_{[S]} \mid S \text{ simple}\}$.

We denote by $\mathcal{H}_{\mathbf{Q}}(R) = \mathcal{H}(R) \otimes_{\mathbf{Z}} \mathbf{Q}$ and $\mathcal{C}_{\mathbf{Q}}(R) = \mathcal{C}(R) \otimes_{\mathbf{Z}} \mathbf{Q}$ the rational extension of the Hall–Ringel algebra $\mathcal{H}(R)$ and composition algebra $\mathcal{C}(R)$, respectively.

Now we have the following consequence.

PROPOSITION 2.1. *Let $L, M_1, M_2, \ldots, M_l \in \operatorname{mod} R$ and $(M_1, M_2, \ldots, M_l)$ be a factor sequence of L. Then*

$$l_{\mathbf{Z}}\big(\mathrm{Ext}^1_R(L, L)\big) \le l_{\mathbf{Z}}\left(\mathrm{Ext}^1_R\left(\bigoplus_{i=1}^{l} M_i, \bigoplus_{i=1}^{l} M_i\right)\right)$$

and we have strict inequality if $L \not\cong \bigoplus_{i=1}^{l} M_i$.

Proof. We use induction on l. The case $l = 2$ follows from Lemma 2.1. For $l > 2$, there is a submodule U of L with $U \simeq M_l$ such that L/U has a factor sequence $(M_1, M_2, \ldots, M_{l-1})$. By induction we have

$$\begin{aligned} &l_{\mathbf{Z}}\big(\mathrm{Ext}^1_R(L, L)\big) \\ &\quad \le l_{\mathbf{Z}}\big(\mathrm{Ext}^1_R(L/U \oplus M_l, L/U \oplus M_l)\big) \\ &\quad = l_{\mathbf{Z}}\big(\mathrm{Ext}^1_R(L/U, L/U)\big) + l_{\mathbf{Z}}\big(\mathrm{Ext}^1_R(L/U, M_l)\big) \\ &\qquad + l_{\mathbf{Z}}\big(\mathrm{Ext}^1_R(M_l, L/U)\big) + l_{\mathbf{Z}}\big(\mathrm{Ext}^1_R(M_l, M_l)\big) \end{aligned}$$

$$\leq l_{\mathbf{Z}}\Big(\mathrm{Ext}^1_R\Big(\oplus_{i=1}^{l-1} M_i,\ \oplus_{i=1}^{l-1}\ M_i\Big)\Big) + l_{\mathbf{Z}}\Big(\mathrm{Ext}^1_R\Big(\oplus_{i=1}^{l-1} M_i, M_l\Big)\Big)$$
$$+ l_{\mathbf{Z}}\Big(\mathrm{Ext}^1_R\Big(M_l,\ \oplus_{i=1}^{l-1}\ M_i\Big)\Big) + l_{\mathbf{Z}}\big(\mathrm{Ext}^1_R(M_l, M_l)\big)$$
$$= l_{\mathbf{Z}}\Big(\mathrm{Ext}^1_R\Big(\oplus_{i=1}^{l} M_i,\ \oplus_{i=1}^{l}\ M_i\Big)\Big).$$

If $L \not\simeq \oplus_{i=1}^{l} M_i$, then $L/U \not\simeq \oplus_{i=1}^{l-1} M_i$ or $L \not\simeq L/U \oplus M_l$ and hence we have strict inequality.

This completes the proof. ∎

Given any R module M and any simple R-module, we denote by $(\underline{\dim} M)_S$ the *Jordan–Hölder* multiplicity of S in M (i.e., the number of the composition factors in a composition series of M which are isomorphic to S. We call the function $\underline{\dim} M$ the *dimension vector* of M. We say that an indecomposable R-module M is uniquely determined by its dimension vector provided that if $\underline{\dim} M = \underline{\dim} N$, then $N \simeq M$, for any indecomposable module N.

Recall that a *short cycle* in R-mod is a pair of indecomposable R-modules together with nonzero nonisomorphisms $f: M \to N$ and $g: N \to M$.

PROPOSITION 2.2 (Reiten, Skowronski, and Smalø [4]). *If* R-mod *has no short cycle, then any indecomposable R-module is uniquely determined by its dimension vector.*

3. THE UNIVERSAL PBW BASES OF HALL–RINGEL ALGEBRAS

Given an finite set J, we denote its cardinality by $|J|$.

Let I be a set, let $\mathsf{P}(I)$ be the set of all finite subsets of I, let $\overline{\mathsf{P}}(I) = \{(J, \mathbf{n}) | J \in \mathsf{P}(I),\ \mathbf{n} \in \mathbf{N}^{|J|}\}$, and let $O(I) = \{(J, \prec) | J \in \mathsf{P}(I)$ and $\prec$ is a linear order on $J\}$. A map α: $\overline{\mathsf{P}}(I) \to O(I)$ will be called a weight-order map if $\alpha(J) = (J, \prec)$ for some linear order $\prec$ of J. Write $(J, \prec) = (a_1, a_2, \dots, a_{m_J})$ if $a_1^J \prec a_2^J \prec \cdots \prec a_{m_J}^J$ under the linear order $\prec$ of J.

DEFINITION 3.1. Let A be an associative algebra over some field K with the identity element 1 and let I be a non-empty subset of A which doesn't contain 1. Then I is a *universal PBW basis* of A if, for any weight-order ordered map α of I,

$$\Big\{a_1^{n_1} a_2^{n_2} \cdots a_{m_J}^{n_{m_J}} | (J, \mathbf{n}) \in \overline{\mathsf{P}}(I), \mathbf{n} = (n_1, n_2, \dots, n_{M_J}),$$
$$\alpha(J) = (a_1, a_2, \dots, a_{m_J})\Big\} \cup \{1\}$$

is a K-basis of A.

Now we discuss the existence of the universal PBW basis in the Hall–Ringel algebras.

We introduce the following partial order on the set of the isomorphism classes of modules in R-fin. Let $\delta(M) = l_{\mathbf{Z}}(\mathrm{Ext}^1_R(M, M))$.

THEOREM 3.1. *$I = \{u_{[X]} | X \in R$-fin indecomposable$\}$ is a universal PBW basis of $H_{\mathbf{Q}}(R)$.*

Proof. For any $0 \neq M \in R$-fin, let

$$J(M) = \left\{u_{[X]} | [X] \text{ is an indecomposable summand of } M\right\};$$

it is a finite subset of I. If $M \simeq \oplus_{i=1}^m n_i X_i$ for $i \in \mathbf{N}$, let $\mathbf{n}_M = (n_1, \dots, n_m)$. Fix an order $J(M) = (u_{[X_1]}, u_{[X_2]}, \dots, u_{[X_m]})$. Then $\alpha: (J(M), \mathbf{n}_M) \to (u_{[X_1]}, u_{[X_2]}, \dots, u_{[X_m]})$ defines a weight-order map on I. By Proposition 2.1, we have that if $\delta(M) = 0$, then

$$u^{n_1}_{[X_1]} u^{n_2}_{[X_2]} \cdots u^{n_m}_{[X_m]} = a_M u_{[M]}$$

and if $\delta(M) > 0$, then

$$u^{n_1}_{[X_1]} u^{n_2}_{[X_2]} \cdots u^{n_m}_{[X_m]} = a_M u_{[M]} + \sum_{[M] \succ [N]} a_N u_{[N]},$$

where $a_N = F^L_{X_1, \dots, X_m}$ for $\delta(N) < \delta(M)$, by Proposition 2.1. Clearly $a_M \neq 0$ is invertible in $\mathbf{Q}$. Since $\{u_{[M]} | M \in R\text{-fin}\}$ is a Q basis of $H_{\mathbf{Q}}(R)$, from the above formulas we see that

$$\left\{u_{[0]} = 1\right\} \cup \left\{u^{n_1}_{[X_1]} u^{n_2}_{[X_2]} \cdots u^{n_{m_J}}_{[X_{m_J}]} | M \simeq \oplus_{i=1}^m n_i X_i \text{ for } i \in \mathbf{N},\ M \in R\text{-fin} \right.$$
$$\left. \times \alpha\left(J(M), \mathbf{n}_M\right) = \left(u[X_1], \dots, u[X_m]\right)\right\}$$

is $\mathbf{Q}$-linear independent and using induction with respect to $\delta(M)$ for $M \in R$-fin we have that $u_{[M]}$ is a linear combination of

$$\left\{u^{n_1}_{[X_1]} u^{n_2}_{[X_2]} \cdots u^{n_{m_J}}_{[X_{m_J}]} | (J, \mathbf{n}) \in \overline{P}(I), \alpha(J, \mathbf{n}) = \left(u_{[X_1]}, \dots, u_{[X_{m_J}]}\right)\right\}$$

for any $0 \neq M \in R$-fin. Therefore

$$\left\{u^{n_1}_{[X_1]} u^{n_2}_{[X_2]} \cdots u^{n_{m_J}}_{[X_{m_J}]} | (J, \mathbf{n}) \in \overline{P}(I), \alpha(J, \mathbf{n}) \right.$$
$$\left. = \left(u_{[X_1]}, \dots, u_{[X_{m_J}]}\right)\right\} \cup \left\{u_{[0]} = 1\right\}$$

is a $\mathbf{Q}$ basis of $H_{\mathbf{Q}}(R)$. This proves that I is a universal PWB basis of $H_{\mathbf{Q}}(R)$. ∎

4. HALL–RINGEL ALGEBRAS AND COMPOSITION ALGEBRAS

THEOREM 4.1. *If* R-mod *has no short cycle, then* $\mathsf{H}_{\mathbf{Q}}(R) = \mathcal{C}(R)$.

Proof. We prove that $u_{[M]}$ is in $\mathcal{C}(R)$ for all M in R-fin, using induction on the length $l(M)$ of the R-module M.

The assertion follows trivially for M with $l(M) = 1$. Assume that $1 < l(M)$ and that $u_{[N]} \in \mathcal{C}(R)$ for all N in R-fin with $l(N) < l(M)$. We first suppose that M is decomposable and $M = N_1 \oplus N_2$ with N_1 indecomposable. If there exists an indecomposable R-module X with $\underline{\dim} X = \underline{\dim} M$, then we have that $\mathrm{Hom}_R(X, N_1) = 0$, or $\mathrm{Hom}_R(N, X_1) = 0$, since there is no short cycle in R-mod. We may assume that $\mathrm{Hom}_R(X, N_1) = 0$. Then $F^X_{N_1, N_2} = 0$. If such an X does not exist, then $F^Y_{N_1, N_2} = 0$ for any indecomposable R-module Y since any indecomposable R-module is uniquely determined by its dimension vector by Proposition 2.2. So we may assume that L is decomposable if $F^L_{N_1, N_2} \neq 0$. We will prove by induction on $\delta(M) = l_{\mathbf{Z}}(\mathrm{Ext}^1_R(M, M))$ that $u_{[M]} \in \mathcal{C}_{\mathbf{Q}}(R)$.

If $\delta(M) = 0$ then $u_{[N_1]}u_{[N_2]} = F^M_{N_1, N_2} u_{[M]}$ and $F^M_{N_1, N_2} \neq 0$. So by the inductive assumption on the length we have that $u_{[N_1]}, u_{[N_2]} \in \mathcal{C}_{\mathbf{Q}}(R)$ and hence $u_{[M]} \in \mathcal{C}_{\mathbf{Q}}(R)$. Let $0 < \delta(M)$. Then we have

$$u_{[N_1]}u_{[N_2]} = F^M_{N_1, N_2} u_{[M]} + \sum_{[L]} F^L_{N_1, N_2} u_{[L]},$$

where L ranges through all isoclasses of decomposable R-modules L with $\underline{\dim} L = \underline{\dim} M$ and $\delta(L) < \delta(M)$. By inductive assumption all such $u_{[L]}$ are in $\mathcal{C}_{\mathbf{Q}}(R)$. By the inductive assumption on the length we get $u_{[N_1]}u_{[N_2]} \in \mathcal{C}_{\mathbf{Q}}(R)$. So $u_{[M]} \in \mathcal{C}_{\mathbf{Q}}(R)$ since $F^M_{N_1, N_2} \neq 0$.

Now we suppose that M is indecomposable and let M' be a submodule of M where $M/M' = S$ is simple. Then

$$u_{[S]}u_{[M']} = F^M_{S, M'} u_{[M]} + \sum_{[L]} F^L_{S, M'} u_{[L]},$$

where L ranges through all isoclasses of decomposable R-modules since any indecomposable R-module N with $F^N_{S, M'} \neq 0$ is isomorphic to M by Proposition 2.2. We have just proved that all such $u_{[L]}$ are in $\mathcal{C}_{\mathbf{Q}}(R)$. By the inductive assumption on the length, we get $u_{[M']} \in \mathcal{C}_{\mathbf{Q}}(R)$. So $F^M_{S, M'} \neq 0$ implies that $u_{[M]} \in \mathcal{C}_{\mathbf{Q}}(R)$.

This completes the proof. ∎

5. HALL POLYNOMIALS

In this section, we consider the existence of Hall polynomials for some algebras R of finite representation type. In particular, we will prove that any representation-finite trivial extension algebra has Hall polynomials. We also prove that if R-mod has no short cycle, then R has Hall polynomials. Throughout this section, we assume that k is a finite field and R is a locally bounded k-algebra, that is, R is an associative algebra and R has a set of primitive orthogonal idempotents $\{e_i\}_I$ such that $R = \oplus_{i,j\in I} e_i R e_j$, and both $\dim_k Re_i$ and $\dim_k e_i R$ are finite for all $i \in I$. It is easy to see that R is finitary and R-mod $= R$-fin.

Let E be a field extension of k. For any k-space V, we denote by V^E the E-space $V \otimes_k E$. Of course, R^E naturally becomes an E-algebra. We recall from [9] that E is *conservative* for R if for any indecomposable $M \in \operatorname{mod} R$, $(\operatorname{End} M/\operatorname{rad}\operatorname{End} M)^E$ is a field. R *has Hall polynomials* provided that for any $X, Y, Z \in \operatorname{mod} R$, there exists a polynomial $\varphi^Y_{Z,X} \in \mathbf{Z}[T]$ such that for any conservative finite field extension of k for R,

$$\varphi^Y_{Z,X}(|E|) = F^{Y^E}_{Z^E, X^E}.$$

Such a φ^Y_{ZX} is called a *Hall polynomial.*

Let $R = kQ$, where k is a prime field and Q is a quiver of type A_2. Then any finite field extension of k is a conservative field of R.

Let $\Omega_R = \{$finite field extension E of $k | E$ is conservative for $R\}$. Note that if R is representation-finite, then Ω is an infinite set.

If R is representation-finite and if R has Hall polynomials, then for any $M, N_1, \dots, N_t \in R$-mod, there is a polynomial $\varphi^M_{N_1,\dots,N_t} \in \mathbf{Z}[T]$ such that for any $E \in \Omega_R$,

$$\varphi^M_{N_1,\dots,N_t}[|E|] = F^{M^E}_{N_1^E,\dots,N_t^E}.$$

This follows from the fact that for any $1 \le i \le t$, we have

$$F^{M^E}_{N_1^E,\dots,N_t^E} = \sum_{[L]} F^{M^E}_{N_1^E,\dots,N_{i-1}^E, L^E} F^{L^E}_{N_i^E,\dots,N_t^E},$$

where $[L]$ goes through the isoclasses of all $L \in R$-mod. In this case, we also call $\varphi^M_{N_1,\dots,N_t}$ a *Hall polynomial.*

By a similar discussion as above, we have the following (see also [3])

PROPOSITION 5.1. *Let $\mathbf{C}$ be a full subcategory of* R-mod *closed under isomorphisms and assume that R is representation-finite. If for any $X, Y, Z \in$* R-mod *with $X \in \mathbf{C}$, R has the Hall polynomials $\varphi^Y_{X,Z}$ and $\varphi^Y_{Z,X}$, then for any $M \in$* R-mod *and $N_1, \dots, N_t \in$* R-mod *with at most one of the N_i's not in $\mathbf{C}$, R has the Hall polynomials $\varphi^M_{N_1,\dots,N_t}$.*

The following two propositions are proven in [3].

PROPOSITION 5.2. *Assume that R is representation-finite. Then for any $M, N, S \in \operatorname{mod} R$ with S simple, R has the Hall polynomials $\varphi_{S,N}^{M}$ and $\varphi_{N,S}^{M}$.*

Let $A, B, X \in R$-mod. We denote by $\operatorname{Ext}_R^1(B, A)_X$ the set of exact sequences in $\operatorname{Ext}_R^1(B, A)$ with the middle term X, by $W(A, B; X)$ the set

$$\{(f, g) \in \operatorname{Hom}_R(A, X) \times \operatorname{Hom}_R(X, B) \mid f \text{ injective, } g \text{ projective, } fg = 0\},$$

and by $V(A, B; X)$ the set of all the filtrations $X \supseteq U \supseteq 0$ of X such that $U \simeq A$ and $X/U \simeq B$.

PROPOSITION 5.3. *For any $X, A, B, \dots, R$-mod, we have*

$$F_{B,A}^{X} = \frac{|\operatorname{Ext}_R^1(B, A)_X|\,|\operatorname{Aut}_R X|}{|\operatorname{Aut}_R A|\,|\operatorname{Aut}_R B|\,|\operatorname{Hom}_R(B, A)|}.$$

In the proofs of the following results, we will need the following lemma. Its proof is easy; see, for example, [6].

LEMMA 5.1. *Let $\varphi, \psi \in \mathbf{Z}[T]$. And assume that ψ is monic. Then ψ divides φ if and only if the integer $\psi(q)$ divides the integer $\varphi(q)$ for infinitely many $q \in \mathbf{Z}$.*

PROPOSITION 5.4. *For any $M, N \in R$-mod, there exists a monic Hall polynomial $\varphi_{M,N}^{M\oplus N}$.*

Proof. For any $E \in \Omega_R$, by Proposition 5.2, we have

$$F_{M,N}^{M\oplus N} = \frac{|\operatorname{Ext}_{R^E}^1(M^E, N^E)_{M^E\oplus N^E}|\,|\operatorname{Aut}_{R^E}(M^E \oplus N^E)|}{|\operatorname{Aut}_{R^E} M^E|\,|\operatorname{Aut}_{R^E} N^E|\,|\operatorname{Hom}_{R^E}(M^E, N^E)|}.$$

By Lemma 2.1 we have $\operatorname{Ext}_{R^E}^1(M^E, N^E)_{M^E\oplus N^E} = 0$. Following Proposition 1.1 of [3] there exist monic polynomials $\gamma_{M,N}, \alpha_M, \alpha_N, \alpha_{M\oplus N} \in \mathbf{Z}[T]$ such that

$$\gamma_{M,N}(|E|) = |\operatorname{Hom}_{R^E}(M^E, N^E)|, \qquad \alpha_M(|E|) = |\operatorname{Aut}_{R^E} M^E|,$$
$$\alpha_N(|E|) = |\operatorname{Aut}_{R^E} N^E|, \qquad \alpha_{M\oplus N}(|E|) = |\operatorname{Aut}_{R^E} M^E \oplus N^E|.$$

Denote $\varphi_{M,N}^{M\oplus N} = \alpha_{M\oplus N}/\alpha_M \alpha_N \gamma_{M,N}$. By Lemma 5.1 we see $\varphi_{M,N}^{M\oplus N} \in \mathbf{Z}[T]$ and hence it is a Hall polynomial. Clearly $\varphi_{M,N}^{M\oplus N}$ is monic. This completes the proof. ∎

Now we obtain some results about the existence of Hall polynomials for some algebras.

THEOREM 5.1. *Let R be representation-finite and assume that for any $Y, X_1, X_2 \in R$-mod with X_1 indecomposable, R has Hall polynomials $\varphi^Y_{X_1X_2}$ and $\varphi^Y_{X_2X_1}$. Then R has Hall polynomials.*

Proof. By Proposition 5.1, for any $Y \in R$-mod and $X_1, X_2, \dots, X_s \in \operatorname{ind} R$, R has the Hall polynomial $\varphi^Y_{X_1, X_2, \dots, X_s}$.

For any $M, M_1 \in M_t \in R$-mod with all $M_i \neq 0$, we will prove by induction that $\varphi^M_{M_1, \dots, M_t}$ exists.

Put $l(M) = l$, the length of the R-module M.

If $l = 0$, then take $\varphi^M_{M_1, \dots, M_t} = 0$. So we fix $l > 0$ and assume that for any $s \geq 0$ and $N, N_1, \dots, N_s \in R$-mod with $l(N) < l$, $\varphi^N_{N_1, \dots, N_s}$ exists.

Suppose that $t = l$. If there is some $E \in \Omega_R$ such that $F^{M^E}_{M_1^E, \dots, M_t^E} \neq 0$, then all M_i must be simple and hence $\varphi^M_{M_1, \dots, M_t}$ exists. If $F^{M^E}_{M_1^E, \dots, M_t^E} = 0$ for all $E \in \Omega_R$, we take $\varphi^N_{N_1, \dots, N_s} = 0$. So we can assume that $l > t$ and for $s > t$ and $N_1, N_2, \dots, N_s \in R$-mod with all $N_i \neq 0$, $\varphi^M_{N_1, \dots, N_s}$ exists.

If all the M_i are indecomposable, then we know that $\varphi^M_{M_1, \dots, M_t}$ exists. So we assume that for any $X \in \operatorname{ind} R$, $\varphi^M_{M_1, \dots, M_{i-1}, X, M_{i+1}, \dots, M_t}$ exists and that some $M_i = H_1 \oplus H_2$ with $H_1 \neq 0$ and $H_2 \neq 0$. We use induction on $\delta(M_i) = l_{\mathbf{Z}}(\operatorname{Ext}_R(M_i, M_i))$.

If $\delta(M_i) = 0$, then for any $E \in \Omega_R$, we have that $\operatorname{Ext}^1_{R^E}(M_i^E, M_i^E) = 0$ and that by Proposition 2.2, $F^{L^E}_{H_1^E, H_2^E} \neq 0$ implies $L^E \simeq M_i^E$. Therefore

$$F^{M^E}_{M_1^E, \dots, M_i^E, \dots, M_t^E} F^{M_i^E}_{H_1^E, H_2^E} = F^{M^E}_{M_1^E, \dots, M_{i-1}^E H_1^E, H_2^E, M_{i+1}^E, \dots, M_t^E}.$$

By Proposition 5.3 there is a monic Hall polynomial $\varphi^{M_i}_{H_1, H_2}$ and by the above assumption there is the Hall polynomial $\varphi^M_{M_1, \dots, M_{i-1}, H_1, H_2, M_{i+1}, \dots, M_t}$. So by Lemma 5.1, $\varphi^M_{M_1, \dots, M_i, \dots, M_t}$ exists.

If $\delta(M_i) > 0$, then for any $E \in \Omega_R$, we have

$$F^{M^E}_{M_1^E, \dots, M_i^E, \dots, M_t^E} F^{M_i^E}_{H_1^E, H_2^E} = F^{M^E}_{M_1^E, \dots, M_{i-1}^E, H_1^E, H_2^E, M_{i+1}^E, \dots, M_t^E} - \sum_{[L]} F^{M^E}_{M_1^E, \dots, M_{i-1}^E, L^E, M_{i+1}^E, \dots, M_t^E} F^{L^E}_{H_1^E, H_2^E},$$

where the sum takes over all $[L]$ with $L \in R$-mod and $\delta(L) < \delta(M_i)$. In case L is indecomposable, we see that $\varphi^M_{M_1, \dots, M_{i-1}, L, M_{i+1}, \dots, M_t}$ exists by the above assumption. In case L is decomposable, we see by induction that $\varphi^M_{M_1, \dots, M_{i-1}, L, M_{i+1}, \dots, M_t}$ exists also. And by the above assumptions, both $\varphi^M_{M_1, \dots, M_{i-1}, H_1, H_2, M_{i+1}, \dots, M_t}$ and $\varphi^L_{H_1, H_2}$ exist. Note that we have a monic Hall polynomial $\varphi^{M_i}_{H_1, H_2}$. Therefore by Lemma 5.1, $\varphi^M_{M_1, \dots, M_i, \dots, M_t}$ exists. This completes the proof. ∎

We recall that if A is an algebra over some field k', denoted by D the dual functor $\mathrm{Hom}_{k'}(-, k')$, then DA has a canonical structure of the $A-A$-bimodule, and the trivial extension algebra of A, denoted by $A \ltimes DA$, is a k'-algebra whose basic k'-space is $A \oplus DA$, the direct product of k'-spaces A and DA, whose multiplication is defined by

$$(a, \varphi)(b, \psi) = (ab, a\psi + \varphi b)$$

for any $a, b \in A$ and $\varphi, \psi \in DA$.

In [3], it has been shown that if R belongs to the class of representation-finite trivial extension algebras or to the class of algebras defined in the following corollary, then for any $Y, X_1, X_2 \in R$-mod with X_1 indecomposable, R has the polynomial $\varphi^Y_{X_1 X_2}$ and $\varphi^Y_{X_2 X_1}$. Using the above theorem, we hence obtain the following.

COROLLARY 5.1.1. *If R is representation-finite and there is a locally bounded k-algebra A which is directed, such that there exists a covering functor F: A-mod $\to R$-mod, and for any $M, N \in R$-mod there exist $X, Y \in \mathsf{A}$-mod with $FX = M$ and $FY = N$ such that F induces the k-isomorphism*

$$\mathrm{Ext}^1_{\mathsf{A}}(X, Y) \xrightarrow{\sim} \mathrm{Ext}^1_R(M, N),$$

then R has Hall polynomials. In particular, if R is a representation-finite trivial extension algebra, then R has Hall polynomials.

THEOREM 5.2. *Assume that R-mod has no short cycle. Then R has Hall polynomials.*

Proof. Since R-mod has no short cycle, R is representation-finite and any indecomposable R-module is uniquely determined by its dimension vector.

By Propositions 5.1 and 5.2, for any $Y, X_1, X_2, \dots, X_s \in R$-mod with all X_i simple, R has the Hall polynomial $\varphi^Y_{X_1, X_2, \dots, X_s}$.

For any $M, M_1, \dots, M_t \in R$-mod with all $M_i \neq 0$, we will prove by induction that $\varphi^M_{M_1, \dots, M_t}$ exists.

Put $l = l(M)$, then length of M.

If $l = 0$, then take $\varphi^M_{M_1, \dots, M_t} = 0$. So we fix $l > 0$ and assume that for any $s \geq 0$ and $N, N_1, \dots, N_s \in R$-mod with $l(N) < l$, $\varphi^N_{N_1, \dots, N_s}$ exists.

If all the M_i are simple, then we know that $\varphi^M_{M_1, \dots, M_t}$ exists. So we assume that some M_i are not simple.

We first assume that M is decomposable and suppose that $M_i = H_1 \oplus H_2$ with H_1 indecomposable. If there exists an indecomposable R-module X with dimension vector $\underline{\dim}\, X = \underline{\dim}\, M_i$, then we have that $\mathrm{Hom}_R(H_1, X) = 0$ or $\mathrm{Hom}_R(X, H_1) = 0$ since R-mod has no short cycle. We may assume that $\mathrm{Hom}_R(X, H_1) = 0$; then $F^X_{H_1, H_2} = 0$. If such an X

does not exist, then $F^{Y}_{H_1, H_2} = 0$ for the indecomposable module Y since the indecomposable modules are uniquely determined by their dimension vectors. So we may assume that Y is decomposable if $F^{X}_{H_1, H_2} \neq 0$. We now use induction on $\delta(M_i) = l_{\mathbf{Z}}(\mathrm{Ext}_R(M_i, M_i))$.

If $\delta(M_i) = 0$, then for any $E \in \Omega_R$, we have that $\mathrm{Ext}^1_{R^E}(M_i^E, M_i^E) = 0$ and that by Proposition 2.1, $F^{L^E}_{H_1^E, H_2^E} \neq 0$ implies $L^E \simeq M_i^E$. Therefore

$$F^{M^E}_{M_1^E, \dots, M_i^E, \dots, M_t^E} F^{M_i^E}_{H_1^E, H_2^E} = F^{M^E}_{M_1^E, \dots, M_{i-1}^E, H_1^E, H_2^E, M_{i+1}^E, \dots, M_t^E}.$$

By Proposition 5.4 there is a monic Hall polynomial $\varphi^{M_i}_{H_1, H_2}$ and by the above assumption there is the Hall polynomial $\varphi^{M}_{M_1, \dots, M_{i-1}, H_1, H_2, M_{i+1}, \dots, M_t}$. So by Lemma 5.1, $\varphi^{M}_{M_1, \dots, M_i\ M_t}$.

If $\delta(M_i) > 0$, then for any $E \in \Omega_R$, we have

$$\begin{aligned} & F^{M_E}_{M_1^E, \dots, M_i^E, \dots, M_t^E} F^{M_i^E}_{H_1^E, H_2^E} \\ &\quad = F^{M^E}_{M_1^E, \dots, M_{i-1}^E, H_1^E, H_2^E, M_{i+1}^E, \dots, M_t^E} \\ &\qquad - \sum_{[L]} F^{M^E}_{M_1^E, \dots, M_{i-1}^E, L^E, M_{i+1}^E, \dots, M_t^E} F^{L^E}_{H_1^E, H_2^E}, \end{aligned}$$

where the sum takes over all $[L]$ with $L \in R$-mod decomposable and $\delta(L) < \delta(M_i)$. So $\varphi^{M}_{M_1, \dots, M_{i-1}, L, M_{i+1}, \dots, M_t}$ exists by induction, and by the above assumptions both $\varphi^{M}_{M_1, \dots, M_{i-1}, H_1, H_2, M_{i+1}, \dots, M_t}$ and $\varphi^{L}_{H_1, H_2}$ exist. Note that we have a monic Hall polynomial $\varphi^{M_i}_{H_1, H_2}$ by Proposition 5.4. Therefore by Lemma 5.1, $\varphi^{M}_{M_1, \dots, M_i, \dots, M_t}$ exists.

We then suppose that M_i is indecomposable. Put $H_1 = \mathrm{top}\, M_i \neq 0$ and $H_2 = \mathrm{rad}\, M_i \neq 0$. Note that M_i is unique determined by its dimension vector. So

$$\begin{aligned} & F^{M^E}_{M_1^E, \dots, M_i^E, \dots, M_t^E} F^{M_i^E}_{H_1^E, H_2^E} \\ &\quad = F^{M^E}_{M_1^E, \dots, M_{i-1}^E, H_1^E, H_2^E, M_{i+1}^E, \dots, M_t^E} \\ &\qquad - \sum_{[L]} F^{M^E}_{M_1^E, \dots, M_{i-1}^E, L^E, M_{i+1}^E, \dots, M_t^E} F^{L^E}_{H_1^E, H_2^E}, \end{aligned}$$

where the sum takes over all $[L]$ with $L \in R$-mod decomposable. We have just proved that $\varphi^{M}_{M_1, \dots, M_{i-1}, L, M_{i+1}, \dots, M_t}$ exists. By the above assumptions both $\varphi^{M}_{M_1, \dots, M_{i-1}, H_1, H_2, M_{i+1}, \dots, M_t}$ and $\varphi^{L}_{H_1, H_2}$ exist. Note $F^{M_i^E}_{H_1^E, H_2^E} = 1$. Therefore $\varphi^{M}_{M_1, \dots, M_i, \dots, M_t}$ exists. This completes the proof. ∎

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REFERENCES

1. J. A. Green, Hall algebras, hereditary algebras and quantum groups, *Invent. Math.* **120** (1995), 361–377.
2. J. Y. Guo, The calculation of the Hall polynomials for a cyclic quiver, *Comm. Algebra* **23,** No. 2 (1995), 743–751.
3. L. Peng, Some Hall polynomials for representation-finite trivial extension algebras, preprint, SFB94053, Universität Bielefeld, 1994.
4. I. Reiten, A. Skowronski, and S. O. Smalø, Short chain and short cycle of modules, *Proc. Amer. Math. Soc.* **117** (1993), 343–354.
5. C. M. Ringel, Tame algebras and integral quadratic forms, *in* "Lecture Notes in Mathematics," Vol. 1099, Springer-Verlag, New York/Berlin, 1984.
6. C. M. Ringel, Hall algebras, *in* "Topics in Algebras," pp. 443–447, Banach Centre Publ., Vol. 26, PWN, Warsaw, 1990.
7. C. M. Ringel, Hall algebras and quantum groups, *Invent. Math.* **101** (1990), 583–592.
8. C. M. Ringel, Lie algebras arising in representation theory, *in* "London Math. Soc. Lecture Note Ser.," Vol. 168, pp. 284–291, Cambridge Univ. Press, Cambridge, UK, 1992.
9. C. M. Ringel, The composition algebra of a cyclic quiver—Towards an explicit description of the quantum group of type $\tilde{A}_n$, *Proc. London Math. Soc.* (3) **66** (1993), 507–537.